An Exterior Ballistics Primer

with

Applications to 9mm Handguns

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2014
revised 2018

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Motivation

I first developed an acquaintance with guns as an Indiana teenager. Starting with the boy's standard first rifle, a .22 caliber Remington used mostly for target practice and sometimes for hunting, I added a .22 Ruger revolver and, as a collector's item, an 1878 Sharps .45-70 rifle like the one later highlighted in the movie *Quigley Down Under*— a lethally gorgeous very long range rifle; regrettably, doubts about its integrity and the lack of ammunition kept me from firing it. When I enlisted in the Marine Corps in 1961, these guns were sold (I hope they're not on the Detroit streets, especially the Sharp) and guns became an occupational tool. At that time I learned the difference between my rifle and my gun, the only lasting knowledge my drill instructor passed on.

In the early 1960s the USMC used the clunky and reliable M1 Garand Rifle as its training weapon. The M1 had been replaced by the M14—virtually identical except with a magazine—as a field weapon in 1957, and the M14 was soon to be replaced by the M16—the workhorse of Viet Nam and the prototype of the civilian AR15 that haunts school hallways today. In Advanced Training after Boot Camp, I was required to "qualify" with the M1 at intervals of 250, 300, and 500 yards after a period of zeroing in at 100 yards. The result could be Marksman, Sharpshooter, or Expert. My first attempt was a dismal failure marked by rapid waves of Maggie's Drawers—the red flag that target managers wagged to inform you of a complete miss; I didn't even achieve Marksman status. My drill instructor, remarkably excited by this deficiency, sent me to the ophthalmologist, where I discovered the meaning of myopia. After being fitted out with the pink-rimmed glasses *du jour* that the Corps issued then (someone in Purchasing had a sense of humor), I returned to the firing line and qualified as an Expert.

After the Corps I had no contact with guns: there was no time for them, and they were not socially acceptable for an economics professor teaching in Massachusetts. But on retiring and moving to the high-caliber-friendly state of Florida, I re-established my acquaintance. The initial reason was that I developed a love of the water and bought a boat. There are places you don't want to be in a boat without defensive equipment, so I acquired two Walther 9mm pistols (a P99 and a PPS) and a Mossberg 500 shotgun; and I returned to target practice.

Now, one can do target practice on the cheap from a boat: simply throw an object overboard (choose one that floats) and bang away. This has the advantages that no other shooters are around to damage your ears, and that you learn to shoot on an unstable platform. But it can create safety concerns for other boaters. As a result, I became interested in the question. "What is the maximum range of my 9mm handguns?"

This is a matter of "Exterior Ballistics," the forces on and motions of a bullet after it exits the barrel. Initially I simply looked up the answer—about 2,300 yards. But then the question took on a life of its own—in my many years of research I've repeatedly encountered instances where the published word was wrong because mistakes crept in. So I decided to see for myself. This primer is the result of that research. I hope someone finds it useful.

The following analysis uses some specific language. The units of measurement are in the English PFS (pound-foot-second) units rather than the Metric KMS (kilogram-meter-second) units most commonly used in physics. Addendum 1 defines the units and the fundamental definitions used here.

Words like "range" and "distance" refer to horizontal distance, "slant range" is the direct line-of-sight distance to a target when it is above or below the shooter. The word "velocity" refers to velocity along the bullet's path (tangent to the arc at the bullet's position); when referring to the up-down or in-out motion we refer to "vertical velocity" or "horizontal velocity." "Mass" and "weight" are synonyms unless otherwise stated (as in atmospheric density measurements where the pound-mass (lb_m) is used to specifically indicate mass).

A note for those less mathematically inclined. I have laid out the details of the mathematical analysis, and after the first equation your eyelids will begin to flutter. Not to worry: just ignore the math and scan the text. There will still be a lot to learn about basic exterior ballistics.

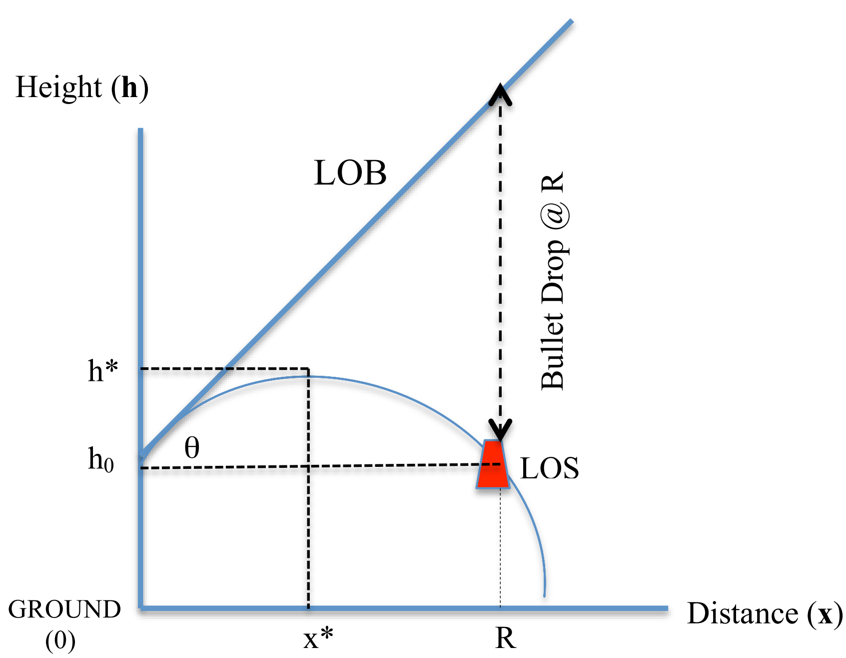
If you see a need for corrections or have any polite suggestions, please let me know at *webmaster@fortunearchive.com*.

Exterior Ballistics in a Vacuum

When a bullet is fired the trigger releases a hammer or a firing pin that slams into the base of the cartridge, where a small amount of highly explosive matter—the *primer*—is stored. The primer's explosion ignites the powder just forward of the cartridge base and the ignited powder releases gases that expand rapidly and push the bullet out of its cartridge, accelerating the bullet from dead stop to perhaps 4,000 feet per second in just a few inches. The pressure of the released gases as they exit the muzzle pushes on air molecules and creates a "blast." If the muzzle velocity is supersonic (greater than about 1,125 feet per second) a pressure wave is created and the blast is accompanied by the "crack" of a sonic boom.

The activity of the bullet before it exits the muzzle is a matter of the "Interior Ballistics" of the weapon. What happens once the bullet is released is the focus of "Exterior Ballistics." We begin with Figure 1 demonstrating the parameters underlying the analysis, the Marksman's Setup shown in Figure 1.

Figure 1
The Marksman's Setup



The analysis that follows is in three stages: "first approximation" that is, ballistics in a vacuum with no gravity; "second approximation," that is, ballistics in a vacuum with gravity; and "third approximation," that is, ballistics in an atmosphere with both gravity and aerodynamic drag.

First Approximation: Bullet Path in a Vacuum

A projectile is launched at velocity v_0 feet/second at an angle of $\theta°$ (the launch angle) from a height h_0 feet above the ground to hit a target at the shooter's elevation and at horizontal distance R (the range). In this picture the Line-of-Sight (LOS) from shooter to target is horizontal. The shooter elevates his weapon so that the Line-of-Bore (LOB) exceeds the LOS. We know he does this because of gravity, but for the moment let's ignore both gravity and air resistance ("drag").

In the absence of gravity the bullet will proceed forever along the LOB at constant velocity v_0 (the "muzzle velocity"). Obviously the target will be missed—after t seconds of travel the bullet will be at distance v_0t along the LOB, always above the LOS. If the shooter really thought there was no gravity he should choose a zero-degree *angle of reach*, aiming at the target directly along the LOS.

The bullet's velocity can be decomposed into vertical motion at velocity v_h and horizontal motion at velocity v_x, both of which are constant. These velocities will conform to the equation $v_0 = \sqrt{(v_h^2 + v_h^2)}$. It turns out that the vertical and horizontal velocities will be $v_h = v_0 sin\theta$ and $v_x = v_0 cos\theta$ so the height of the projectile along the LOB is $h(t) = h_0 + v_0(sin\theta)t$ and the projectile's horizontal distance on the LOB is $x(t) = v_0(cos\theta)t$.

Second Approximation: Bullet Path in a Vacuum with Gravity

When gravity is introduced the trajectory is no longer a straight line—it is a parabola, an arc along which the height of the bullet and any horizontal distance x is the height of the LOB at x less the bullet drop at x due to gravity. The bullet drop is at any moment t after launch is $– ½gt^2$ so the bullet's height is now $h(t) = h_0 + v_0t – ½gt^2$. This can be converted to a drop at distance x by using the fundamental equation relating t and x, that is, $x(t) = v_0(\cos\theta)t$. Note that the projectile's velocity on the arc (at any point tangent to the arc) remains v_0 throughout the bullet's path, but the division between velocity's vertical and horizontal components changes along the flight path.[1]

This information is shown in Figure 1 above. There are two forces operating on the projectile. The first is the force of the *initial propulsion*; this sends the projectile along the LOB starting at h_0, traveling at muzzle velocity v_0 at angle θ. At any time after firing, the distance traveled along the LOB line is v_0t, the height above the shooter is $h(t) = (v_0 \sin\theta)t$ and the horizontal distance from the shooter is $(v_0 \cos\theta)t$.

The height of the bullet above the ground and the horizontal distance travelled are

(1) a. $$h(t) = h_0 + (v_0 sin\theta)t - \frac{1}{2}gt^2$$

b. $$x(t) = v_0(\cos\theta)t$$

Using these fundamental equations we can write height as a function of horizontal distance travelled:

(1') $$h(x) = h_0 + (\tan\theta)x - ½[g/(v_0\cos\theta)^2]x^2$$

The maximum height (also called "maximum trajectory" or "maximum ordinate") of the bullet, denoted as h^*, occurs at time t^* and horizontal distance x^*, described by the three equations in system (2)[2]

(2) a. $$x^* = \frac{1}{2}(v_0)sin2\theta$$

b.[3] $$h^* = h_0 + (tan\theta)x^* - \frac{1}{2}[\frac{g}{(v_0 cos\theta)^2}]x^{*2}$$

c. $$t^* = \frac{x^*}{(v_0 cos\theta)}$$

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[1] Recall that force equals mass times acceleration, i.e. F = ma. For an object subject to gravity and a gravitational constant of g we have F = -mg. Eliminating mass from both sides we get simply a = -g. Mass "disappears" and gravitational force tells us that for a falling body *acceleration* is negative (i.e., in the downward direction); the bullet's acceleration is a constant determined solely by the gravitational force g—the bullet's mass doesn't matter.

[2] The maximum height occurs when $dh/dx = 0$, (or, equivalently, $dh/dt = 0$). Readers not acquainted with basic calculus can directly solve equation (1') for its two roots using the quadratic formula. Equation (2a) employs the identity $\sin\theta\cos\theta = ½ \sin2\theta$.

The trajectory also has *terminal conditions*—values of time, distance, energy, velocity, and so forth at the moment when the bullet hits the target. The range (R) and the time (T) of impact are essential to this analysis. To find the range, substitute R for x in equation (1') and find the value of R for which $h(R) = 0$; this will be the solution to the quadratic equation (3a). Once R is found, T can be found by setting x = R in equation (1b); this gives the terminal time as shown in equation 3b.

(3) a. $0 = h_0 + (tan\theta)R - \frac{1}{2}[\frac{g}{(v_0 cos\theta)^2}]R^2$

b. $T = R/(v_0 cos\theta)$

Finally, we have the concept of "bullet drop," to be distinguished from the "bullet path" (also called the "bullet trajectory)." Bullet path is the arc taken by the bullet and it includes distances above and below the shooter—when shooting at an upward angle the bullet path numbers begin as positive numbers because they are above the LOS, then they shift to negative numbers as gravity pulls the bullet down below the shooter's line of sight. Bullet drop is often obtained from manufacturer's information based on standard conditions. It is also available in ballistics software used to adjust the launch angle θ so the shooter can "come up" or "come down" the correct amount when the distance to the target differs from the distance for which the weapon is zeroed. The bullet drop at distance x is the vertical distance from the point on the LOB at distance x to the bullet's height at that distance (see Figure 1 above for bullet drop at range R). Denoting the bullet drop as D(x), the equation for bullet drop at each distance x is

(4) $D(x) = (v_0 sin\theta)x - h(x)$

that is, the drop at distance x is the height of the LOB *less* the bullet's height above the LOS at that distance.

Conversely, if you know the launch angle θ and the bullet drop $D(x)$ at any range, you can use (4) to calculate the bullet's height at that range. Drop is usually measured in inches, though we will use yards and convert to inches when convenient. Minutes of Angle (MOA) and Mil-dots are also used to measure drop.[4]

It is important to emphasize that this scenario excludes any consideration of "drag," that is, air resistance due to atmospheric or other factors creating drag (air density, temperature, humidity, turbulence at the projectile, pitch and yaw of the projectile). Drag will dramatically reduce the maximum height, range, and both the average and terminal velocity of the projectile.

Addendum 2 summarizes the equations of motion used in exterior ballistics calculations without drag.

Two important aspects of this exercise are noteworthy—we will alter these when we consider ballistic drag.

- *The mass of the projectile does not enter into the bullet's motion. The reason is that in the absence of drag, mass affects neither the velocity along the flight path nor (as Galileo showed) the gravitational characteristics encountered by the projectile.*

- *The velocity of the projectile is always equal to the muzzle velocity, v_0. However, velocity is decomposed into vertical velocity (v_h) and horizontal velocity (v_x) using the equation $v = \sqrt{(v_x^2 + vy_h^2)}$. Along the projectile's trajectory the composition of velocity changes but velocity itself remains constant.*

Table 1 below reports the results for our sample bullet used throughout this study: a 124 grain 9mm Luger with a muzzle velocity of 1,110 feet per second. For these calculations we have set the shooter's elevation at ground level and equal to the elevation of the target (that is, $h_0 = h_1 = 0$); the spreadsheets below show the results at different launch angles.

The range-maximizing launch angle is 45° at which the maximum range of over 12,755 yards (7.25 miles) is achieved and the maximum height of 9,566 yards (5.4 miles) is reached in 24½ seconds; and the bullet impacts the ground in 49 seconds. At a 90° launch angle, when the bullet is shot straight up and all of its motion is vertical, the bullet reaches a height of almost 7 miles, travels no horizontal distance, and returns to Earth in

[4] Minute of Angle (MOA) is a measure of the number of degrees in the vertical rotation of the LOB: One MOA is 1/60 of a degree and translates to a 1" impact shift per 100 yards of distance to the target. A Mil-dot is a military unit equal to (roughly) one inch over a 100-yard distance.

69 seconds. Note that the bullet's path is a perfect parabola: it leaves at launch angle θ°, it peaks at horizontal distance $x^* = \frac{1}{2}R$, and it hits the earth at distance R = $2x^*$ at an impact angle θ°.

Most shooters don't want to set a launch angle and calculate the range. Instead, they want to find the launch angle that makes the bullet hit a target at a specified horizontal and vertical point; this launch angle is called the *Angle of Reach*. For example, zeroing a rifle to hit the center of a target 6 feet tall at, say, 200 yards, requires finding the launch angle that leads to impact at the target's middle—three feet above ground—at a distance of exactly 200 yards. Having found that angle of reach, the sights are set so that the LOB is elevated above the direct line of sight (LOS) at exactly that angle. The shooter can then put the crosshairs directly on a target 200 yards away, knowing that the launch angle is adjusted correctly to hit the target. A shot at any other distance will require some sight readjustment.

Table 2 reports the angle of reach at different horizontal distances for our sample bullet. There are two launch angles that will reach any given distance: the higher launch angle, exceeding 45°, allows the bullet to arch high and then descend to the target. The lower launch angle, below 45°, gives a gentler parabolic arc upward toward the target. For example, our bullet will hit a target ½-mile distant using a launch angle of either 88.0° or 2.0°.

Don't rush out and use these data unless you are operating in a vacuum with no drag. Your chances of hitting a target 7¼ miles away using a 9mm Luger bullet are absolutely zero—when drag resistance is factored in, that target is *way* out of range.

Table 1

Bullet Trajectories at various Launch Angles, without Drag

Sample Bullet: MagTech 124 Grain Full Metal Jacket 9mm Luger

Data

h_0	0	**is shooter's height relative seal level** *(feet)*
v_0	1,110	**is muzzle velocity** *(ft/sec)*
g	32.2	**is the gravitational constant** *(ft/sec/sec)*
h_1	0	**is height of the target relative to sea level** *(feet)*

Launch Angle		Range and Flight Time				Maximum Height States				Terminal States (at time T)		
		Range	Range	Range	Flight Time	Maximum Height	Time of Max Height	Distance at Max Height	Velocity *	Horizontal Velocity	Vertical Velocity	Angle of Impact
(Degrees)	*(Radians)*	*(feet)*	*(yards)*	*(miles)*	*(seconds)*	*(feet)*	*(seconds)*	*(feet)*	*(ft/sec)*	*(ft/sec)*	*(ft/sec)*	*(degrees)*
-25	-0.4363	0	0	0	0	0	0.00	0	1,110	1,110	0	-25.00
-20	-0.3491	0	0	0	0	0	0.00	0	1,110	1,110	0	-20.00
-15	-0.2618	0	0	0	0	0	0.00	0	1,110	1,110	0	-15.00
-10	-0.1745	0	0	0	0	0	0.00	0	1,110	1,110	0	-10.00
-5	-0.0873	0	0	0	0	0	0.00	0	1,110	1,110	0	-5.00
0	0.0000	0	0	0	0	0	0.00	0	1,110	1,110	0	0.00
5	0.0873	6,644	2,215	1.26	6.01	145	3.00	3,322	1,110	1,106	97	5.00
10	0.1745	13,087	4,362	2.48	11.97	577	5.99	6,544	1,110	1,093	193	10.00
15	0.2618	19,132	6,377	3.62	17.84	1,282	8.92	9,566	1,110	1,072	287	15.00
20	0.3491	24,596	8,199	4.66	23.58	2,238	11.79	12,298	1,110	1,043	380	20.00
25	0.4363	29,312	9,771	5.55	29.14	3,417	14.57	14,656	1,110	1,006	469	25.00
30	0.5236	33,138	11,046	6.28	34.47	4,783	17.24	16,569	1,110	961	555	30.00
35	0.6109	35,956	11,985	6.81	39.54	6,294	19.77	17,978	1,110	909	637	35.00
40	0.6981	37,683	12,561	7.14	44.32	7,905	22.16	18,841	1,110	850	713	40.00
45	0.7854	38,264	12,755	7.25	48.75	9,566	24.38	19,132	1,110	785	785	45.00
50	0.8727	37,683	12,561	7.14	52.81	11,227	26.41	18,841	1,110	713	850	50.00
55	0.9599	35,956	11,985	6.81	56.48	12,838	28.24	17,978	1,110	637	909	55.00
60	1.0472	33,138	11,046	6.28	59.71	14,349	29.85	16,569	1,110	555	961	60.00
65	1.1345	29,312	9,771	5.55	62.48	15,715	31.24	14,656	1,110	469	1,006	65.00
70	1.2217	24,596	8,199	4.66	64.79	16,894	32.39	12,298	1,110	380	1,043	70.00
75	1.3090	19,132	6,377	3.62	66.59	17,850	33.30	9,566	1,110	287	1,072	75.00
80	1.3963	13,087	4,362	2.48	67.90	18,555	33.95	6,544	1,110	193	1,093	80.00
85	1.4835	6,644	2,215	1.26	68.68	18,987	34.34	3,322	1,110	97	1,106	85.00
90	1.5708	0	0	0.00	68.94	19,132	34.47	0	1,110	0	1,110	90.00

*** Note: Velocity is constant at v_0 in the absence of drag, though the mix between vertical and horizontal components changes as the bullet travels**
Mass does not affect the trajectory in the absence of drag

Source: http://en.wikipedia.com/wiki/Trajetory_of_a_projectile

Table 2

Angle of Reach at Various Horizontal Distances

Distance-Feet		**660**	**1,320**	**2,640**	**5,280**	**10,560**	**15,840**	**21,120**	**26,400**	**31,680**	**36,960**
Yards		**220**	**440**	**880**	**1,760**	**3,520**	**5,280**	**7,040**	**8,800**	**10,560**	**12,320**
Miles		**1/8 mile**	**1/4 mile**	**1/2 mile**	**1 mile**	**2 miles**	**3 miles**	**4 miles**	**5 miles**	**6 miles**	**7 miles**
Angle	**-- High**	89.5	89.0	88.0	86.0	82.0	77.8	73.2	68.2	62.1	52.5
	-- Low	*0.5*	*1.0*	*2.0*	*4.0*	*8.0*	*12.2*	*16.8*	*21.8*	*27.9*	*37.5*

Source: http://en.wikipedia.com/wiki/Trajetory_of_a_projectile

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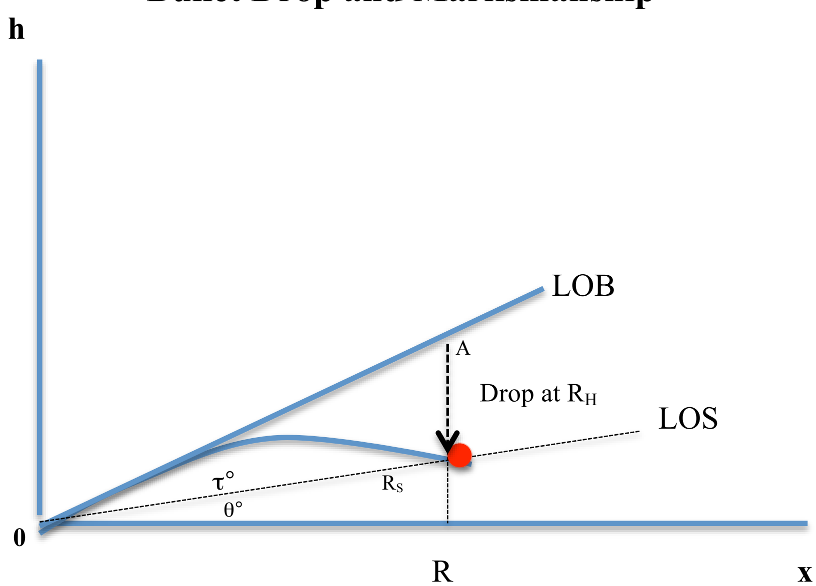
Shooting on a Slope: The Rifleman's Rule

Most shooters spend their time shooting on nearly level ground at targets either at rest or moving at the same elevation, as is shown in Figure 1. Under those circumstances, two things are noteworthy: first, the distance along the LOS (the *Slant Range,* denoted as R_S) is equal to the horizontal distance to the target (the *Range*, denoted as R); the reason is, of course, that the angle of reach is zero forcing R_S and R to be equal.

But what about a shooter who is above or below his target? There is a scene in the movie *True Grit* when Rooster Cogburn (John Wayne or Jeff Bridges, depending on the version) shoots from a high mesa downward at a sharp angle to hit a horseman riding slowly along on the plain far below. This was well before the iPhone and ballistics apps would do the calculations for him. Even so, he hits the target—it's a movie—and I'm sure that most folks in the audience were impressed, perhaps applauding at such fine marksmanship. But if they knew the difficulties of shooting on a slope, they would have awarded Rooster a Nobel Prize in Marksmanship.

Figure 2
Bullet Drop and Marksmanship



In Figure 2 above the target is at horizontal distance R but the target is elevated above the shooter; The LOS is tilted upward at angle θ. The distance to the target along the LOS is called the *slant range* and is denoted by R_S. The shooter's task is to determine the "come-up" angle, τ, by which he must elevate the LOB above the LOS angle to make the bullet drop at range R exactly the value that intersects the LOS at the target. Knowing the drop at R, he then aims directly at point A, allowing gravity to bring the bullet down to the target.

Consider a simple situation where the slant range is 300 yards and the target is elevated 100 yards relative to the shooter—a real Rooster Cogburn shot. The old-fashioned way to address this was to consult a bullet drop table to compute the distance below the LOB the bullet will be at the target's range. For example, a 150-grain .30-06 bullet will drop about 23 inches at 300 yards and about 20 inches at 285 yards (normal atmospheric conditions). The shooter who recognizes that "range" and "slant range" are very different concepts will compute the correct target line-of-sight angle (it is 19.47º) and then calculate the range as $R_S \cdot \cos(19.47°)$, about 283 yards. He will shoot so that the bullet drops by 20 inches at 283 yards, intersecting the bull's eye on the target.

Suppose the shooter confuses the slant range with the range. He notes that the slant range is 300 yards but forgets that the horizontal range is less than that—and that it is horizontal range that determines both the time to impact and the associated horizontal range. Thus, he correctly sees the LOS angle as 19.47º but thinks the target is 17 yards farther out on the LOS. Consulting his drop table he will find a 23-inch drop at 300 yards so he will aim for a point 23 inches above the bulls eye. The true drop at 283 yards is 20 inches, not 23 inches, so he will hit the target at a point 3 inches higher than his point of aim. For many purposes this isn't a problem, but for a hunter a hit 3 inches high could be the difference between a clean kill and a long walk after a wounded animal.

The fundamental Rifleman's Rule is simple: *Don't be a Dope, Shoot Low on a Slope.* But by how much should the shooter compensate? The exact answer lies in some more trigonometry, now provided by hand-held ballistics computers. Anecdotal evidence suggests that less experienced hunters, having heard the Rifleman's Rule, tend to overcompensate, coming down too much and shooting low. This suggests that experience is the answer to correct implementation of the rule.

Consider a bullet fired at a zero-degree launch angle (called "flat-fire"). The only vertical force on the bullet is gravity and the drop at any moment is - $\frac{1}{2}[g/(v_0 cos\theta)^2]x^2$ or $-\frac{1}{2}gt^2$, depending on whether you want to relate the drop to elapsed time or horizontal distance. As we see later in Table 6, drag induces lower bullet drop as LOB angle increases, so shooters most prone to shooting high are those who confuse slant range with horizontal range *and* use flat-fire bullet drop data that overstates drop.

A more accurate Rifleman's Rule, called the *Improved Rifleman's Rule*, makes a simple scale adjustment to the slant range: the slant range is multiplied by some fraction to derive the correct distance (range) for bullet drop. The correct scale adjustment is to multiply the slant range by the cosine of the angle-of-sight along the LOS; in our example, this gives 300cos(19.47°), or 282.84 yards. We've seen this number before—it is the target's actual horizontal range! The scale adjustments made are given in Table 3. Thus, at a 20° LOS angle the 300-yard slant rage is multiplied by .940 to give a range of 282 yards; the precise adjustment factor for 19.47° is .9428, for a range of 282.84 yards.

Table 3

Slant Range and Cosine Adjustment

Measured Line of Sight Angle	Corresponding Cosine Figure
10°	0.985
15°	0.966
20°	0.940
25°	0.906
30°	0.866
35°	0.819
40°	0.766
45°	0.707
50°	0.643
55°	0.574
60°	0.500

Many shooters are not versed in trigonometry, some perhaps not in multiplication. For them the Rifleman's Rule is sometimes stated in a looser form. For example, John Plaster, a retired Army sniper, suggests a version gleaned from the FBI practice book: if the LOS angle is between 30° and 45°, aim for 90% of the slant range; if it is below 30°, aim for 70% of the slant range; if it is below 10° shoot flat. This applies regardless of the slope's direction.

Yet another variation uses a bullet drop table for the particular ammunition used; this table is typically provided by the manufacturer. It's steps are: first, calculate the LOS angle θ and the horizontal range as $R_S \bullet cos(\theta)$; second, determine from the bullet manufacturer's published drop tables the flat-fire bullet drop at the target's horizontal range; and finally, calculate the come-up angle as $\tau = arctan\,[\frac{-\Delta(R)}{R}]$.[5] This can be modified to become $\tau = arctan\,[\frac{-(\cos\theta)\Delta(R_S)}{R_S}]$ as the come-up. Of course, by the time you've done the calculation the target has moved on!

[5] Drop Tables are reported in inches. Brian Litz's *Point Mass Ballistics Solver* gives a drop of 114.71 inches, or 3.19 yards, at a 280 yard range

The Effect of Atmospheric Drag

We have discussed exterior ballistics in a vacuum with gravity; now we look at the bullet path when atmospheric drag is added. This is our third approximation model of ballistics. We also discuss arcane force-like effects like the Coriolis Effect and the Magnus Effect that can affect projectiles over long distances but are not an integral part of our third approximation model.

<u>Forces and Effects inducing Ballistic Drag</u>

Ballistic drag—deceleration of a projectile while in flight—and other "forces" or "effects"—dramatically alter a projectile's trajectory. Exterior Ballistics—the forces affecting the projectile after launch—combine with other factors to make projectile dynamics extremely complex

• *The bullet's shape:* drag is related to the shape of the bullet, particularly its cross-sectional area (the area of the circle that the bullet presents to it flight path) and the shape at its stern. Bullets with larger cross-sectional area encounter greater air resistance and greater deceleration; bullets with stern shapes like a boat-tail generate less wake turbulence and lower resistance than flat-based bullets; bullets with pointed noses have less resistance than flat nosed bullets like hollow-points.

• *The bullet's mass:* given its cross-sectional area, greater mass reduces bullet drag as mass punches through the drag. Mass is typically measured in grains, once associated with the weight of a grain of wheat: 7,000 grains per Imperial pound. A common 9mm bullet's mass is in the range 115-147 grains (7.5 – 9.5 grams); the WWII M1 Garand's ,30-0 cartridge had a bullet weighing 150 grains. .45 caliber bullets can run up to 400 grains.

• *The bullet's stability:* a bullet in flight does not stay exactly "on target." Rather, it is subject to yaw (horizontal wobble), pitch (vertical wobble), and precession (circular nose wobble due to both pitch and yaw). If these are too great the bullet can tumble in flight, greatly increasing its drag. To counter these, the bore is rifled to cause the bullet to rotate in flight, creating angular velocity that reduces wobble. The rifling is calculated to achieve an optimal "twist rate" at which the stabilizing properties of rotation and its energy-using effects are balanced. Every gun is designed with this in mind. For example, the four-inch barrel on a Walther P99 induces a twist rate of one rotation every ten inches of travel; this is the standard "twist rate" for handguns.

• *Muzzle velocity:* This is particularly important when it exceeds the speed of sound, about 1,125 feet per second. At supersonic velocities air resistance and drag are considerably greater than at subsonic velocities. As a bullet advances, its velocity falls, so if the muzzle velocity is supersonic, drag decreases sharply as it becomes subsonic. Muzzle velocity is typically determined by barrel length and by the quantity and quality of the powder used as a propellant. The M-16 military rifle has a 5.56x45mm bullet. This is only .22 caliber bullet, the same as the standard match or varmint pistol, but the M-16 bullet is longer (more stable), more massive, and, propelled through a long barrel by more propellant; it achieves a muzzle velocity of 3,110 feet per second, much higher than the standard .22 caliber bullet or than the long-retired M1 Garand bullet. The 150-grain M2 Ball cartridge used in the Garand remains supersonic for more than 1,000 yards so I never gets to the transonic zone after which drag drops sharply.

• *Air density:* air density is a complex combination of air pressure, altitude (typically lower drag at higher altitudes), humidity (greater drag at higher water vapor content per unit of air), and air temperature (higher temperature generally means lower density and less drag).

A variety of other "effects" affect a bullet's path. The *Coriolis Effect* is related to the Earth's rotation. It is the reason that winds moving northward from the equator arch toward the east in the northern hemisphere. Suppose that a shooter is interested in a target at higher latitude (say, he is on the equator aiming directly north toward Chicago). As the Earth rotates, Chicago will always be directly north of his position, that is, the longitude remains unchanged. But, because of the Earth's spherical shape, his speed of travel (rotational velocity) is greater than Chicago's: at the equator he is rotating (relative to, say, the sun) at 480 meters per second or approximately 1,000 mph, but Chicago, at latitude 41° 51'N, is rotating at roughly 500 mph. In effect, the shooter must recognize that his bullet will travel eastward relative to Chicago because Chicago is moving westward relative to him; he must *lag* the target. In contrast, a Chicago shooter aiming at our equatorial shooter must *lead* his target. Clearly, the Coriolis Effect is important for long-range bullets, particularly ballistic missiles. But it can be of some concern for long-range rifles as well.

Every golfer knows the *Magnus Effect*: a "slice" sends the ball veering to the right because the ball is rotating clockwise relative to the vertical plane, reducing the air pressure in the direction of the spin; the differential pressures on the right and left of the flight path create lift that pushes the ball to the right. While the professional golfer can control that motion, the shooter cannot—bullet rotation is required for stability. For example, a Walther 9mm pistol has a right-handed rotation as it exits the barrel. This right-handed spin is relative to the vertical plane, and, like the sliced golf ball, the bullet is deflected to the right. The response of a professional shooter is to adjust for this, aiming a bit leftward to compensate.

Measuring Ballistic Drag: The Ballistic Coefficient

A central concept in ballistic drag analysis is the "standard" bullet, used to calibrate the drag of a specific "sample" bullet. To measure the drag coefficient of a sample bullet, a standard bullet is chosen whose drag we know. Then the sample bullet's drag is measured and the ratio of the two drags, called the "form factor," is calculated. This form factor is used in measuring a sample bullet's *ballistic coefficient*.

The earliest model of ballistic drag was based on the "G1" definition of the standard bullet, developed in 1881 by Krupp, the German arms manufacturer, and used, with modifications, by James Ingalls in his seminal 1883 book on exterior ballistics. Though very simple by the standard of modern ballistics, the G1 model is a reasonable standard for pistols using flat-based subsonic bullets. Other models are more appropriate for long-range boat-tailed ammunition; the drag coefficient of supersonic military bullets is typically drawn from a G6 or G7 standard bullet.

At present there are at least six standard bullets, labeled G1 through G8 (there is no G3 or G4); other standards exist but are infrequently used. Each is based on a different set of standard bullet dimensions. For example, the G1 standard bullet (the original Krupp-Ingalls standard) is a flat-based cylindrical bullet 3.28 inches long with a one-inch diameter, a weight of one pound (454 grams), and an "ogive" (sharp nose) having a .02 inch "meplat" (nose-point diameter). By definition, the G1 bullet's "sectional density" is 1.0 and its drag scale factor is 1.0, giving it a ballistic coefficient (defined below) of 1.0.

The G1 serves as the standard for short-range handguns. The most frequently used alternative, used for long-range rifles, is the G7 standard bullet. The G7 is 4.28 inches long, weighing one pound with a one-inch diameter, with a boat-tail stern .6 inches long angled inward at 7.5°, and a slightly blunted nose with a .13 inch meplat; it also has, by definition, a ballistic coefficient of 1.0. In fact, *all* standard bullets are designed with unit values for their BC.

The Ballistic Coefficient of a bullet is essential to the bullet's ballistics. Developed in 1870, the BC measures the bullet's *resistance to* air resistance, so a higher value means less air resistance and, therefore, less drag. A bullet's BC is based on its Sectional Density (SD) and its Drag Scale Factor (DSF). These are[6]

(6) a. $$SD = \frac{m}{d^2}$$

b. $$DSF = \frac{SD}{SD_G}$$

where *d* is the sample bullet's maximum diameter in inches, *m* is its mass (assumed equal to weight) in pounds, SD is the "sectional density" of the sample bullet[7] and SD_G is the sectional density of the standard bullet. Because both numerator and denominator are in the same units (pounds per square inch, or kilograms per square meter), the drag scale factor (DSF) is a pure number. As noted above, sectional density is greater the higher the mass (a heavy bullet resists drag better than a light bullet), and is lower the greater the area facing the flight path (greater area means greater air resistance).

A sample bullet's Ballistic Coefficient is defined as

(7) $$BC = \frac{SD}{f}$$

Note that the form factor, denoted by *f*—has been introduced. This is a direct measure of the bullet's drag coefficient *relative to* that of the standard bullet. In fact, *f* is defined as c_S/c_G, the ratio of the sample bullet's drag coefficient to the standard bullet's drag coefficient.

Ingalls' original work assumed that all sample bullets were scale models of the G1 standard bullet; this is equivalent to setting $f = 1$ so BC is entirely defined by sectional

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[6] SD is a measure of mass relative to the cross-sectional area that the bullet presents to its flight path. The cross-sectional area is properly measured as $\pi d^2/4$, but for historical reasons it is measured as simply d^2 in ballistics calculations.

density. But it soon became obvious that actual drag results differed from Ingalls' predictions because actual bullets rarely conformed precisely to the G1 shape. The solution was to introduce a factor (*f*) to capture these differences: each bullet comes with its own form factor, which, in turn, depends on the specific standard bullet used. For example, the same bullet will have a higher form factor when compared to a "sleek" G7 model than when compared to a "blunt" G1 standard bullet.

The form factor is defined as the ratio of a sample bullet's drag coefficient, c_S, to the standard bullet's drag coefficient, c_G. Substituting this above, we get the following description of the Form Factor:

$$f \equiv c_S/c_G = \frac{SD}{BC} \tag{8}$$

All bullets experience increased drag as velocity increases. Table 4 shows the relationship between a bullet's velocity and its drag coefficients (c_G) for the two primary standard bullets: the G1 and the G7.

The calculations in this document are for a specific sample bullet: a MagTech 124 grain 9mm Parabellum[8] bullet with mass of .0177 pounds, diameter (caliber) of .355, SD = .141 lb./in^2, f = .73 (relative to the G1 bullet), a BC of .192 lb/in^2, and a muzzle velocity of 1,110 feet/sec (at the edge of supersonic). The dimensions of this sample bullet are shown below in Table 5 and its associated image. Some terms need to be defined: the *base diameter* is the width of the narrow base just forward of the cartridge's stern; this is the width of the bullet's stern. The *caliber* is the width of the bullet in inches at the base, where it connects with the cartridge; the *nose diameter* is the width of the *meplat*. The meplat is large for blunt-nosed hollow-point bullets, small for the standard rounded point bullet, or very small for supersonic military bullets.

[8] The 9mm Parabellum is also called the 9x19mm Luger bullet; the first number is the diameter, the second number is the bullet's length. There are other 9mm bullets, for example, the Russian 9x18 Makarov pistol bullet, or the Russian 9x39mm rifle bullet. Georg Luger developed the 9x19mm Parabellumin in 1901 and it became famous as the standard pistol issued to German officers in WWI and WWII. It has become the worldwide bullet of choice for military and police use because it is an excellent compromise between weight, accuracy and stopping power.

The *ogive*, or nose, is the portion of the bullet jutting out from the cartridge; it is called the ogive because it has a curved shape. The ogive length is the length from the tip of the bullet to the cartridge. The ogive radius is a measure of the bullet's pointedness.

Table 4
Velocity and Drag Coefficients

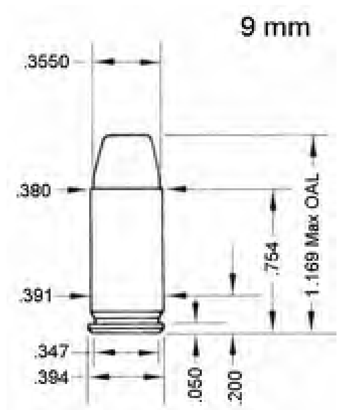
Mach	G1	G7
0.00	.263	.120
0.50	.203	.119
0.60	.203	.119
0.70	.217	.120
0.80	.255	.124
0.90	.342	.146
0.95	.408	.205
1.00	.481	.380
1.05	.543	.404
1.10	.586	.401
1.20	.639	.388
1.30	.659	.373
1.40	.663	.358
1.50	.657	.344
1.60	.647	.332
1.80	.621	.312

Table 5

Dimensions of a 9mm Luger Bullet

		Metric	English
Nose Base Diameter (D)	Width of Bullet at Cartridge (Caliber)	9.03mm	0.355in
Bullet Weight	Mass	124 grains 8.04 grams	. 0177lbs .2863ozs
Ogive Length (L)	Length for Bullet Tip to Cartridge	10.54mm	.415in
Ogive Tip Diameter (T)	Flatness of Nose (Meplat)	.30mm	.012in
Ogive (Nose) Radius (R)	Nose Pointedness	1.67	1.67
Total Length (Nose + Cartridge)	Length of Entire Assembly	29.69mm	1.169 in
Cartridge Length	Length of Unit Holding Propellant	19.15mm	0.754 in

Cartridge Base Diameter	Width of Cartridge Base	8.79mm	0.347in
Dimension	Characteristic	(mm)	(inches)



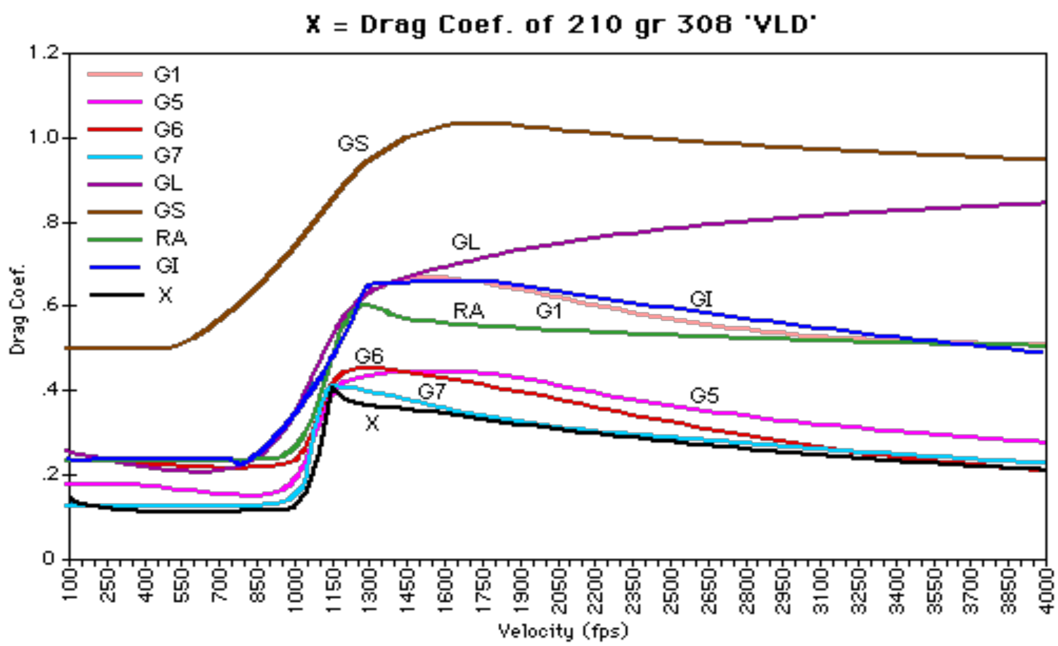
The drag coefficient depends upon all of these dimensions and more. Analytical calculations of drag coefficients are extremely difficult, so most reported drag coefficient values are derived from field measurements by using sound sensors to calculate the velocity decrease between any two points, then comparing that to the velocity decrease for the standard bullet—the ratio of the two is the form factor. If a bullet's velocity decreases by, say, half of the standard bullet's decrease between 250 and 500 yards, then for that interval the form factor for the sample bullet is .50—its drag coefficient is half of the standard bullet's drag coefficient

We've seen that in the absence of drag the only forces affecting a bullet are its initial propulsion and gravity. Throughout the trajectory, velocity remains at muzzle velocity, though the split between vertical and horizontal motion changes. But aerodynamic drag is proportional to the *square* of velocity, so as velocity *decreases* during a bullet's flight, drag decreases exponentially. The decrease in drag is not at a constant rate: an initially supersonic bullet will first experience an *increase* in drag as its velocity slows, but the drag coefficient plunges as the bullet enters subsonic velocities. The slight initial drag increase occurs because the slower-moving bullet loses heat, cooling the air around it and increasing the local air density. The plunge at transonic speeds is because the drag-increasing sonic wave disappears. In contrast, the drag for a subsonic bullet is relatively constant.[9]

This pattern is shown below in Figure 3 for a .308 caliber 210-grain VLD (Very-Low-Drag) rifle bullet, compared to nine standard bullets. For velocities up to 90 percent of the speed of sound (Mach .9) the .308 caliber's drag coefficient rises slightly from .263 to .342, it then jumps to .588 at Mach 1.1, after the transonic range is passed; then it rises slightly to Mach 1.4 and finally falls as Mach 2 is approached. Precise drag analysis requires consideration of this drag vs. velocity relationship.

Figure 3

Drag Coefficients of A Rifle Bullet According to Different Standards



[9] As an object goes supersonic velocities air cannot get out of the way fast enough, creating a pressure increase showing as a pressure ("shock") wave. As the bullet passes, air flows back to fill the vacuum at supersonic speed creating a sonic boom that is heard downrange as the "crack" of the bullet.

Drag, Drop, and Velocity

We saw that in a vacuum a bullet's mass plays no role in its trajectory: the bullet's velocity is constant, though the division between vertical and horizontal velocity changes, and its drop relative to the LOB after t seconds is $-\frac{1}{2}gt^2$. Atmospheric drag changes this: as noted above, as the bullet proceeds its total velocity slows by an amount proportional to the square of velocity; this increases the time taken by the bullet to travel a given distance, thus increasing the time gravity has to work and the drop. It also alters the split between vertical and horizontal velocity.

To demonstrate the effect of drag on bullet drop we have used a readily-available ballistics program to calculate the effects at different launch angles. Table 6 summarizes the results for our Magtech 124-grain 9mm bullet with 1,100 fps muzzle velocity.

Table 6
Bullet Drop with and without Drag
280-yard Horizontal Range

	Bullet Drop		
Launch Angle	**Drag**	**No Drag**	Difference
0°	-137.90"	-110.11"	-27.79"
5°	-137.69"	-109.90"	-27.79"
10°	-136.41"	-108.85"	-27.56"
15°	-134.09"	-106.06"	-27.13"
20°	-130.72"	-104.23"	-26.49"

25°	-126.32”	-100.69”	-25.63”
30°	-120.94”	- 96.36”	-24.58”

Source: Author’s calculations; Brian Litz, *Point Mass Ballistics Solver*

The first column shows the launch angle, the second shows the bullet drop *cum* drag, the third column shows the bullet drop without drag, and the final columns shows the effect of drag on bullet drop[10] We see that at any launch angle drag induces a greater bullet drop, and that drop decreases as launch angle rises.

Because published bullet drop tables play an important role in a shooter’s life, it’s worth investigating the accuracy of the tables, most of which are produced by manufacturers. There are two broad reasons for this question. First, tables are often produced under “flat fire” conditions, which, as Table 6 shows, will give larger drops than shots taken on a slope.

Second, bullet makers are in a competitive environment and a high ballistic coefficient is a marketing tool. Thus, there is a financial incentive to find ways to report a high BC. There is some evidence supporting biased BC’s and drop tables. Courtney and Courtney evaluated the ballistic coefficients reported by several manufacturers for a small sample of bullets, concluding that there was a clear tendency to overstate the BCs (hence understate drag); one bullet’s BC was overstated by twenty-five percent. How might this bias arise? One method is to choose a standard bullet that has high drag, like the G1 bullet; this makes the sample bullet’s drag appear lower when computing the BC.

Another way is to fiddle with atmospheric parameters by, say, basing density estimates on choosing standard atmospheric conditions like ICAO that assume zero humidity instead of “RMY Standard Metro” conditions that assume 78% relative humidity; the dryer air has less density and imparts less drag. In fact, our sample bullet,

[10] Drag is computed using air density of .0747 pounds per cubic feet, consistent with 29.92 inches of mercury barometric press, temperature = 68° and 70 percent relative humidity. With no drag the air density is, of course, zero.

produced by MagTech, might have an overstated BC: MagTech reports it's BC as .192 but 9mm Luger bullets produced by other manufacturers report BCs around .150.

In summary, increased drag increases bullet drop at any horizontal range. There is evidence suggesting that Bullet Drop Tables provided by ammunition manufacturers tend to understate drag and thus understate bullet drop. A shooter relying on published drop tables might well learn to make the Rifleman's Rule adjustments, then shoot even higher to compensate for understated drag.

It is clear that bullet drop is greater with drag. As we shall soon see, the reason is not that drag affects the bullet only in one direction: it affects the bullet in all directions and its effect is the same in all directions. What drag does is to slow the bullet, its rate of rise and fall, and its rate of forward progress. By slowing the bullet's velocity gravity more time to pull the bullet down at any horizontal distance the bullet travels.

To delve more deeply into the effects of drag, we derive the bullet's equations of motion. Aerodynamic drag is arises from a force that is directly proportional to air density and to the square of the projectile's velocity, as seen in equation (9):

$$F_d = -\frac{1}{2} c_G \rho f A v^2 \tag{9}$$

where ρ is air density in pounds per square inch and A is sectional density. Noting that drag deceleration, Δ is $\frac{F_d}{m}$, we can rewrite (9) as (9')

$$\Delta = -\frac{1}{2}\left(c_G \frac{\rho}{BC}\right) v^2 \tag{9'}$$

The negative acceleration (i.e. deceleration) of velocity due to drag is proportional to the square of velocity with the constant of proportionality depending directly on air density and inversely on its ballistic coefficient. Thus, as the velocity along the bullet path diminishes, the drag force diminishes by the *square* of that change—a halving of velocity cuts the drag force by 75 percent. Note that the velocity referred to is measured along the direction of travel, that is, at a point tangent to the bullet's direction of travel on the arc we call its trajectory.

Drag acceleration in two dimensions is described by equation system (10).

(10) a. $\mu = \frac{1}{2}\left(\frac{\rho c_G}{BC}\right)$

b. $\Delta_x = -\mu v v_x$

c. $\Delta_y = -\mu v v_h$

where μ is the *retardation coefficient*, based entirely on air density and bullet characteristics, v is the instantaneous velocity along the direction of travel, v_x is horizontal velocity, and v_h is vertical velocity. Note that μ is a constant while v, v_x, and v_h are functions of time.

Equation system (11) describes the complete system of equations of motion: (11a) defines total velocity as the square root of the sums of squared horizontal and vertical components; (11b) describe horizontal and vertical velocities as functions of the retardation coefficient (μ), of the level and components of velocity (*v, v_x and v_h*) and of the gravitational constant, *g*; Equations (11c) state the initial conditions for the trajectory derived from muzzle velocity (v_0) and launch angle (θ).

a. $v = \sqrt{[v_x^2 + v_h^2]}$

(11) b. $\frac{dv_x}{dt} = -\Delta v v_x \qquad \frac{dv_h}{dt} = -g - \Delta v v_h$

c. $v(0) = v_0 \qquad v_x(0) = (cos\theta)v_0 \qquad v_h(0) = (sin\theta)v_0$

These differential equations have no closed form solution—there is no mathematically exact solution for the trajectory. This is an inherent difficulty in ballistics analysis—even through we know the equations, there is no precise solution. Numerical approximation methods are used to compute discrete approximations to (11). These methods consist of taking the bullet's initial position at the end of the launch and using the equations (11) to update the position in the next nanosecond, then use that second position with (11) to update the calculation to the third nanosecond, and so on.

There are a number of numerical methods that can be used.[11] Euler's Updating Equation is among the most straightforward. Let Δt be a discrete change in time (a "nanosecond") and $F(t)$ be *any* function of time whose first derivative, $F'(t)$ is known. Then the following relationship approximates the exact result:

$$F(t + \Delta t) \approx F(t) + F'(t)\Delta t \tag{12}$$

The approximation improves as Δt becomes smaller, approaching the exact solution as Δt becomes infinitesimally small. Each of the state variables in (11)—*x, h, v*, and so on—can be updated using this equation if given suitable initial conditions.

Before proceeding to our third approximation ballistics model, it is worth noting the contributions of Arthur Pesja to trajectory analysis. Unhappy with the mathematical state of ballistics in the late 1940s, Pesja set out to find a way to make mathematically precise trajectory calculations. To do this he subjected the equations of motion to algebraic approximations that ultimately resulted in a closed form solution that could be easily calculated and that, he believed, were accurate approximations to real data. Pesja's 1953 classic, *Modern Practical Ballistics*, applies line-fitting techniques to empirical drag data to model drag characteristics. A prominent ballistics software package called *ColdBore,* available at Patagonia Ballistics, uses Pesja's methods.

That program is *sui generis*, a *black* black box. Brian Litz says of the Pesja method:

> *The simplicity of the Pesja solution method is only shifted onto the bullet description, which the shooter is burdened to establish for himself for each bullet at all velocities.*
>
> Litz, Bryan. *Applied Ballistics for Long-Range Shooting,* p. 104

The lesson is that the Pesja method—or any method using approximations to obtain closed solutions, does not eliminate complexity, it simply relocates it. Pesja's method requires measurement of bullet characteristics well beyond the ability of normal

[11] A closer approximation often incorporated is ballistics software is the Runge-Kutta method.

shooters. The search for a closed form solution is like shifting deck chairs on HMS Titanic as the iceberg looms—the fundamental problem remains unchanged, only the view changes.

Amanda Wade and others have used Euler's Equation to derive trajectory information. The results are approximations—as are all numerical simulations—but they have several advantages. First, they are transparent—the user knows exactly how the calculations were done; what I call the Euler-Wade model is not a black box where the user has no idea what happens between the inputs and the outputs. Second, the Euler-Wade model is simple and straightforward in its implementation—it can be written as an Excel spreadsheet that can easily be shared and modified as the user sees fit.

We have adapted Wade's approach by modifying her Excel spreadsheet applying Euler's method to (12). The results are reported in the next section. Addendum 3 shows the formulas in the EXCEL spreadsheet. We experimented with the launch angle using the Euler-Wade model, finding that the range-maximizing launch angle for our sample 9mm bullet (MagTech 124-grain) is anywhere between 32° and 34° – at each of those angles the maximum range is 2,311 yards. Note that this is spot-on with the 2,300-yard range reported by the National Rifle Association *Firearms Source Book*, and is consistent with the 2,130-yard range given by the National Law Enforcement and Corrections Technology Institute (See Table 7).[12]

Recall that without drag the maximum range is exactly twice the range at which height is maximum, and the terminal angle of impact is equal to the angle of launch. This is not true with drag. For example, at a 33° launch angle the Euler-Wade model says the maximum height (639 yards) is achieved at a horizontal range of 1,462 yards; this is roughly 63 percent of the 2,311-yard maximum range. Also, the 33° launch angle results in a -64° impact angle. This is consistent with the results of the other ballistics programs used in the next section, *excluding* the *CordBore* program which, by using equations of motion derived from linear approximations to the data, simply repeats the shape of the trajectory implicit in a second-approximation model.

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[12] The NRA and NLECTC sources might have used an Euler-Wade-like model to derive their results; if so, this is not independent verification of Euler-Wade approach.

So How Far *Does* a 9mm Bullet Carry?
Answers from Ballistics Software

The complexity of ballistics models is typically described in terms of the "degrees of freedom" of the model, that is, the number of dimensions of a bullet's path. There are a total of six dimensions, three spatial and three rotational. The three spatial dimensions capture the bullet's flight path—forward-backward, up-down, and right-left—called direction, elevation and drift. The three rotational dimensions are rotational velocity, pitch, and yaw. The most complex model is a 6-DOF mode, while the least complex is a 2-DOF model. For a very stable bullet that doesn't change its attitude as it follows the flight path, a 3-DOF model is "good enough for government work. This, the choice of a 2-DOF model is highly accurate as long as there is no drift.

In the previous section we laid out the basic equations of motion with aerodynamic drag in a 2-DOF ballistics model—the Euler-Wade model. We then used this to simulate the trajectory for our sample 9mm bullet. We found that 2,311 yards was the estimated maximum range. This is consistent with other published reports, but this might not be an independent estimate—if those published reports used the same estimation method—the Euler-Wade model—we have not found independent verification of their conclusions; rather, we have simply found what method they used to get them.

Now we turn to applying a variety of commercial software programs to answer our basic question. Before moving on, it's worth asking not what is the maximum range of a 9-mm bullet but what is the maximum effective range. This, of course, is a question about the shooter as well as about the handgun and bullet used. The longest effective use of a 9mm bullet from a handgun is a mighty 1,000 yards, a shot by Jerry Miculek, who holds world records in pistol marksmanship. This was not in formal competition.

As noted above, advances in computer technology and numerical approximation methods have greatly increased the precision of ballistics calculations. The field is now the province of experts in that technology, among them the late R. L. McCoy, author of *Modern Exterior Ballistics*. This technology has diffused from the military arena to the realm of the laptop computers and PDAs of everyday shooters.

To get the answer to our fundamental question we use the Euler-Wade model, described above, as well as several readily available ballistics software programs, under identical drag-related assumptions (atmospherics, drag coefficients, etc.). We compute trajectories of our sample bullet at several launch angles: 15°, 30°, and 45°, and, where possible, the range maximizing launch angle. Note that the 15° launch angle is the most realistic for handguns: rarely will a pistol shooter want to fire at a higher angle. The other angles are included just to get a sense of their influence on horizontal range.

The ballistics programs we use are all *point mass trajectory calculators*. This means that the bullet is treated as an infinitesimally small point with mass and that rotational dimensions are ignored. In short, they are 3-DOF models in which windage (right-left) is considered as well as forward-backward and up-down. More complicated programs, such as those used for artillery trajectories, treat the shell as an object with both size and rotation. At the top of the ballistics model hierarchy are 6-DOF models that incorporate the three spatial dimensions and three rotational dimensions: spin that causes drift (the Magnus Effect), yaw, and pitch. These models are very accurate and are typically incorporated in analyses of ballistic missile trajectories and in very long range sniper shots.

Consider the following scenario. A shooter uses a 9mm handgun with a four-inch barrel and a ten-inch twist rate—my Walther P99. His load is a MagTech 124 grain Full Metal Jacket round nose bullet with the dimensions reported in Table 6. The shooter is at sea level, the terrain in front of him is flat, and there is no wind. He fires his weapon at a chosen launch angle and calculates the bullet's trajectory as would the program designer—the designer is the man in the black box. After each experiment he gets the results for the entire trajectory—flight time, velocity, energy, height, bullet drop, and horizontal distance—at each moment until impact on the ground. Bullet drop is then used to compute the height of the bullet as the height on the LOB at each range less the bullet drop reported from the black box.

Our shooters in this exercise are:

- The Euler-Wade Model, as modified by this author
- JBM Ballistics, a free online ballistics calculator
- Point Mass Ballistics Solver, a free Mac program developed by Brian Litz

- Sierra Infinity Suite v6, a commercial PC program ($40) from Sierra Bullets
- Ballistics Explorer, a commercial PC program ($70)
- ColdBore, a Pesja-based commercial PC program ($85).

Because ballistics programs are designed to help practical shooters—hunters and target shooters—none are designed to extend out to the far distances involved in calculating a bullet's range to impact at high launch angles (above, say, 15°); after all, a 2,000-yard shot will be an extremely rare event and wildly uncertain. Thus, for some of the programs the longest reported range can end in mid-flight, well before impact or even before maximum height is reached; in this case we record the range as *na*. In the former case, when the reported trajectory ends with a descending bullet, we estimate the range by taking the height and flight angle at the last reported bullet position, estimating an average angle of descent from that point to the ground, calculating the extra yardage to impact, and adding this to the last reported yardage; when this is done the results are marked with †. The parameters used in all of our calculations are reported in Table 7.

Table 7

Parameters Used In Calculations

Symbol	Definition	Value	
m	Bullet Mass	.01773 *lbs*	$8.04\text{x}10^{-3}$ *kg*
d	Bullet Diameter	0.355 *in*	$9.02\text{x}10\text{-}^{3}$*mtr*
A	Cross-Sectional Area,	0.141 in^2	$1.24\text{x}10^{-5} mtr^2$
O <O/d> *	Ogive Radius <O/d>	.196 in <1.67>	4.9784 *mm*
c_{G1}	G1 Drag Coefficient	.22	
f	Relative Drag $(\frac{c_{9mm}}{c_{G1}})$	.73	
c_{9mm}	9mm Drag Coefficient	.1601	
BC	Ballistic Coefficient	.192	
T	Temperature	68F°	21C°
H	Relative Humidity	70%	
P	Air Pressure	29.92 *inHg*	101.325 *kPa (1Atm)*
ρ	Air Density	.07493 lb_m/ft^3	$1.2003 kg/mtr^3$

*The ogive radius O is reported in units of length, but is typically discussed in terms of O/d – length divided by bullet diameter. In the latter form it measures a bullet's pointedness —higher values are more pointed; O/d of .5 is least pointed.

We should point out that JBM Ballistics is the only program that directly calculates maximum range; all others allow the user to do it as we do—by experimenting with the launch angle. There is no other way to address the question of maximum range without letting the software tell its story.

Table 8 reports the results generated by each program. The first column reports the program used, the second reports the launch angle used and (when available) the angle of impact. The final two columns show the maximum height and the range to impact, in yards. For the Euler-Wade model and JBM Ballistics program we also include the results at a range-maximizing launch angle.

The Euler-Wade program is the only program generating results at above 15° for ranges extending out to impact. Thus, it provides impact angles for all launch angles. As expected, these impact angles exceed the launch angles by a substantial margin. The Euler-Wade maximum range results also conform reasonably well to the results of other programs (excluding ColdBore). Because of its transparency, it ease of use, and the high correlation of its results with the other programs, we consider Euler-Wade the most useful of the programs reviewed, at least for purposes of maximum range estimation.

ColdBore claims to be the most comprehensive program: it has a large bullet library, it allows several bullets to be simultaneously compared, it is rich in ballistics information, and its user manual is outstanding. I can not judge how accurate it is for hunters, marksmen, and military shooters, but I can assess how well it performs at the long distances my purpose requires. The answer is, "not very well." *ColdBore*'s distances to impact reported in Table 8 are far greater than found for any other program, and its angle of impact at 15°launch angle is only -16°; this is way too low—as if it dramatically understates the drag coefficient by mimicking the parabolic arch associated with no drag. Perhaps the reason is that it is based on Pesja's work using approximations that might induce greater error at longer distances. Whatever the reasons, we exclude ColdBore from our range assessments, though we leave the ColdBore results in Table 9 so the reader can be aware of the issue.[13]

[13] In communications with Patagonia Ballistics I was warned that ColdBore is designed for military and police use; it is not intended for use at the long ranges in which I am interested. Thus, ColdBore's deficiencies at long ranges are no reason to discount its utility for its intended users. However, if you plan

Table 8
Estimates of Maximum Range
9mm MagTech 124 Grain Bullet

Source	Launch Angle <*Impact Angle*>	Max Height (yards)	Max Distance* (yards)
NRA Sourcebook	24° - 34°	*na*	2,130
NLECTC*	*na*	*na*	2,300
Euler-Wade Solution *(available in this paper)* $0	15° <*-33°*> 30° <*-60°*> ***33°*** **<*-64°*>** 45° <*-74°*>	205 560 **639** 950	1,960 2,305 **2,310** 2,195
Point Mass Solver (Applied Ballistics LLC) $0	15° <*-37°*> 30° < *na* > 45° < *na* >	196 615 1,280	1,820 1,905† 1,855†
JBM Calculator (JBM Ballistics, Online) $0	15° <-37°> 30° < *na* > **35°** **<*-65°*>** 45° < *na* >	200 625 ***na*** *na*	1,850 2,260† **2,250** *na*
Sierra Infinity Suite (Sierra Bullets) $40	15° <-37°> 30° < *na* > 45° < *na* >	230 510 845	1,830 2,035† 2,155†
Ballistics Explorer (Oehler Research) $70	15° <-32°> 30° < *na* > 45° < *na* >	205 715 *na*	2,025 2,725† *na*
ColdBore 1.0 (Patagonia Ballistics) $85	15° <*-16°*> 30° < *na* > 45° < *na* >	217 690 na	2,945 2,990† na

Results in blue font are direct estimates of the maximum range given by the program

* NLECTC is the National Law Enforcement and Corrections Technology Center

† Program does not allow a trajectory to impact. The maximum range is estimated using an estimate of the angle of descent from the bullet's last reported position.

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to imitate the 3,800-yard record for a sniper kill, you should not use ColdBore. And probably none of these programs would serve that need.

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Conclusions

This paper began with a simple question: what is the maximum range of a bullet, particularly a 9mm Luger handgun bullet? To derive an answer we first applied the mathematics of trajectories without drag, in which trajectories are easily calculated as parabolas with maximum range equal to twice the range at which the bullet reaches its maximum height, and with an angle of impact always equals the angle of launch. The maximum range, achieved with a 45° launch angle, is over seven miles, clearly an unrealistic distance even for Rooster Cogburn.

Then we added aeronautical drag to the analysis, delving into the plumbing and wiring behind drag. We reviewed the history of drag-related concepts like drag coefficients and ballistic coefficients, and, having developed the basic foundations, we used several different methods to calculate trajectories of a specific 9mm Luger bullet at various launch angles. A rough summary of the results (excluding ColdBore) is shown below.

Table 9

Summary Estimates of Maximum Range
9mm MagTech 124 Grain Bullet

Launch Angle	Range Min	Range Max	Mean
15°	1790	2025	1865
30°	1905	2725	2225
45°	1850	2200	2005

The first approach, dubbed the Euler-Wade method, uses the equations of motion for projectiles with drag in a 2-DOF space: the two dimensions are vertical and horizontal, there being no wind or other cross-range forces and no consideration of rotational forces. This required inputs for bullet and atmospheric characteristics, as well as initial conditions. These results are embedded in an EXCEL spreadsheet summarized in Addendum 3.

The Euler-Wade method gives a 2,310-yard maximum range, achieved at a 33° launch angle. This is close to the 2,130-yard maximum range reported by the National Rifle Association (at launch angles of 24°-34°), and identical to the 2,300-yard range reported by the National Law Enforcement and Corrections Technology Center,

An advantage of the Euler-Wade method is that we know precisely what goes into the results. This is not true of the results from the use of "black box" ballistics software. We have applied several different ballistics programs to our task: the Point Mass Ballistics Solver, developed by Brian Litz, ballistician for Berger Bullets; the JBM Ballistics Calculator, available online at JBM Ballistics; the Sierra Infinity Suite version 6, a commercial product of Sierra Bullets; and the ColdBore ballistics program from Patagonia Software.

With the exclusion of ColdBore, the results generally confirm those from Euler-Wade. A 9mm Luger-style handgun has a maximum range in windless conditions of about 2,300 yards, or 1.3 miles. And, of course, the range is less at the lower launch angles normally used.

Shoot Away!

Addendum 1

Units of Measurement

Measurement		Metric		English	
Bullet Mass	(μ)	Kilograms	(kg)	Pounds	(lb)
Bullet Diameter	(d)	Millimeters	(m)	Inches	(in)
Form Factor	(*f*)	*No units*		*No units*	
Sectional Density	(SD)	Kilograms/Meter^2	(kg/m^2)	Pounds per Square Inch	(lb/in^2)
Ballistic Coefficient	(BC)	Kilograms/Meter^2	(kg/m^2)	Pounds per Square Inch	(lb/ft^2)
Air Density	(ρ)	Kilograms/Meter^3	(kg/m^3)	Pounds per Square Inch	(lb/ft^3)
Velocity	(v)	Meters per Second	(m/s)	Feet Per Second	(ft/s)
Drag Force	(F_d)	Joules (Kilogram-Meter/^{2}s^2	(J) (kg•m^2/s^2)	Poundal (Pound-Feet/s^2	(ft/s) (lb•ft/s^2)
Retardation Coeff	(Δ)	Meters per Second	(meters/s)	Feet per second	(feet/s)

Unit Conversions

Measurement		Metric Unit		Metric - English
Mass	(μ)	Kilo-gram	(kg)	1 kg = 2.2046 lbs = 35.2740 ozs
Length	(L)	Meter	(m)	1 m = 3.2808 ft = 39.3701 in
Area	(A)	Square Meter	(m^2)	1 m^2 = 10.7639 ft^2 = 1,550.003 in^2
Volume	(V)	Cubic Meter	(m^3)	1 m^3 = 10.7639 ft^3 = 1,550.003 in^3
Pressure*	(P)	Kilo-Pascal	(kPa)	1 kPa = .14504 lb/in^2 = 20.8854 lb/ft^2
Density	(ρ)	Kg per Cubic Meter	(kg/m^3)	1 kg/m^3 = .0624 lb/ft^3 = .00003613 lb/in^3
Velocity	(v)	Meters per Second	(m/s)	1 m/s = 3.2808 ft/s = 38.3701 in/s
Energy	(E)	Joules	(kg•m^2/s^2)	1J = .7376 ft-lbs = 23.7 ft-pdls

* Standard air pressure at sea level is .101325 kPa or 14.696 psi.

Fundamental Definitions

Sectional Density	SD = mass/A
Reference Area	$A = d^2 \;\; \pi d^2/4$
Form Factor (Relative Drag Coefficient)	$f = c_S/c_G$
Ballistic Coefficient	$BC = SD/f$
Drag Coefficient (μ)	$F_d/m = ½[(c_G(\rho/BC)]v^2$

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Addendum 2

The Mathematics of Trajectory in a Vacuum

parameters

h_0 : initial height, in feet, above sea level
h_1 : target's height, in feet, above sea level
v_0 : initial velocity (muzzle velocity) in feet/second
g : gravitational constant (32.2 feet/second/second)
θ : launch angle relative to ground level (equals angle of impact)

State Variables

x : horizontal distance from launch point
t : time in flight
T : time of impact at target; T = time at which h = h_1 and x = R
R : range, horizontal distance to target

Functions

h(x) : height of projectile at distance x
v(x) : velocity at distance x [v(R) is velocity at the target]
θ(x,h)* : Reach Angle, the angle at which the projectile must be launched to hit a target at point *(x,h)*

Equations

Time-Distance Relationship: $x = (v_0 cos\theta)t \qquad t = x/(v_0 cos\theta)$

Maximum Range: $R = (\frac{v_0 cos\theta}{g})\sqrt{}\{v_0 sin\theta + [(v_0 sin\theta)^2 + 2gh_0]\}$

Flight Time: $T = \frac{R}{v_0 cos\theta}$

Height at distance x: $h(x) = h_0 + (tan\theta)x - \frac{1}{2}[\frac{g}{(v_0 cos\theta)^2}]x^2$

Velocity at x: $v(x) = \sqrt{}[(v_v^2) + (v_h^2)]$

Horizontal Velocity at x : $v_h^2 = v_0 cos\theta$

Vertical Velocity at x : $v_v^2 = v_0 sin\theta - \left(\frac{g}{v_0 sin\theta}\right)x$

Launch Angle to hit point (x, h): $\theta^* = \arctan\left(\frac{v_0^2 \pm \sqrt{[v_0^4 - (gx)^2 - 2hgv_0^2]}}{gx}\right)$

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Addendum 3

Euler-Wade EXCEL Spreadsheet
Values View

BULLET TRAJECTORY CALCULATIONS USING EULER UPDATING

Source: Wade, Amanda. *Going Ballistic: Bullet Trajectories*, http://scholarcommons.usf.edu/ujmm/vol4/iss1/5

TO USE THIS PROGRAM INPUT THE DATA IN GREEN CELLS–ALL OTHER DATA ARE DERIVED FROM THOSE

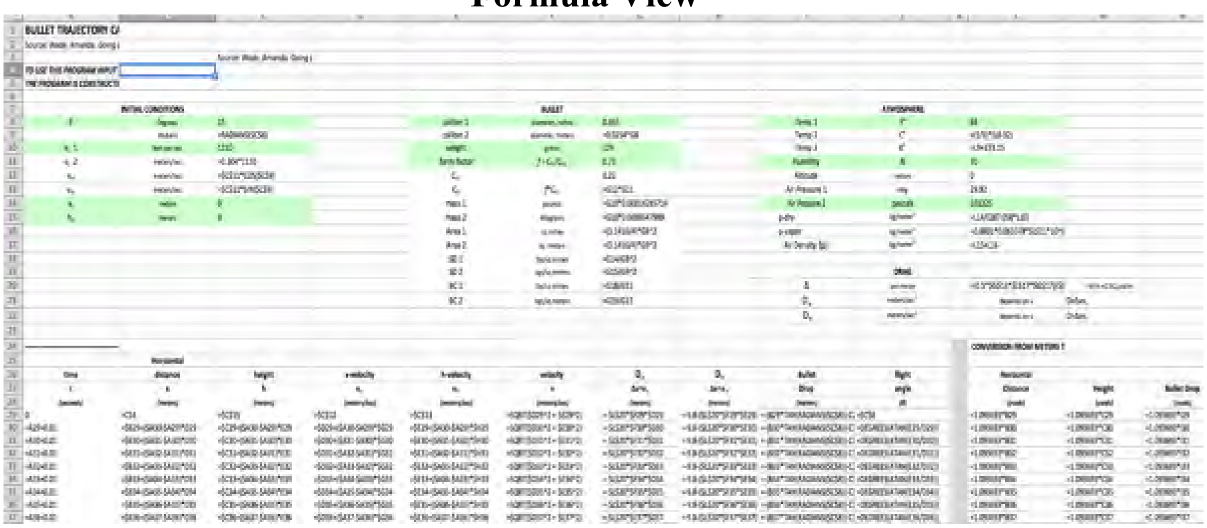
THE PROGRAM IS CONSTRUCTED WITH METRIC DATA

INITIAL CONDITIONS			BULLET			ATMOSPHERE			
θ	Degrees	15.00	caliber 1	diameter, inches	0.3550	Temp 1	F°	68.00	
	Radians	0.26	caliber 2	diameter, meters	0.0090	Temp 2	C°	20.00	
v_0 1	feet per sec	1,110.00	weight	grains	124	Temp 3	K°	293.15	
v_0 2	meters/sec	337.44	form factor	$f = C_x/C_{xx}$	0.7300	Humidity	%	70.00	
v_{x0}	meters/sec	325.94	C_{xx}		0.2200	Altitude	meters	0.00	
v_{h0}	meters/sec	87.34	C_x	$f*C_{xx}$	0.1606	Air Pressure 1	inHg	29.92	
x_0	meters	0.00	mass 1	pounds	0.0177	Air Pesure 2	pascals	101,325.00	
h_0	meters	0.00	mass 2	kilograms	0.0080	p-dry	kg/meter^3	1.20	
			Area 1	sq inches	0.0990	p-vapor	kg/meter^3	0.00	
			Area 2	sq meters	0.0001	Air Density (ρ)	kg/meter^3	1.21	
			SD 1	lbs/sq inches	0.1406				
			SD 2	kgs/sq meters	98.8246		DRAG		
			BC 1	lbs/sq inches	0.1926	Δ	per meter	0.00076931	F d/m=0.5C_xρA/m
			BC 2	kgs/sq meters	135.38	D_x	meters/sec^2	depends on v	D=Δvv$_x$
						D_h	meters/sec^2	depends on v	D=Δvv$_h$

EULER-WADE BALLISTICS MODEL / **CONVERSION FROM METERS TO YARDS**

time	Horizontal distance	height	x-velocity	h-velocity	velocity	D_x	D_h	Bullet Drop	flight angle	Horizontal Distance	Height	Bullet Drop
t	x	h	v_x	v_h	v	Δv*v$_x$	Δv*v$_h$					
(seconds)	*(meters)*	*(meters)*	*(meters/sec)*	*(meters/sec)*	*(meters/sec)*	*(meters)*	*(meters)*	*(meters)*	*(θ)*	*(yards)*	*(yards)*	*(yards)*
0.000	0.00	0.00	325.94	87.34	337.44	-84.61	-32.47	0.00	15.00	0.00	0.00	0.00
0.010	3.26	0.87	325.10	87.01	336.54	-84.17	-32.33	0.00	15.00	3.56	0.96	0.00
0.020	6.51	1.74	324.25	86.69	335.64	-83.73	-32.18	0.00	14.98	7.12	1.91	0.00
0.030	9.75	2.61	323.42	86.37	334.75	-83.29	-32.04	0.00	14.97	10.67	2.85	0.00
0.040	12.99	3.47	322.58	86.05	333.86	-82.85	-31.90	-0.01	14.95	14.20	3.80	-0.01
0.050	16.21	4.33	321.76	85.73	332.98	-82.42	-31.76	-0.01	14.94	17.73	4.74	-0.01
0.060	19.43	5.19	320.93	85.41	332.10	-81.99	-31.62	-0.01	14.92	21.25	5.68	-0.02
0.070	22.64	6.05	320.11	85.09	331.23	-81.57	-31.48	-0.02	14.90	24.76	6.61	-0.02

Formula View



Euler-Wade Equations

Notation

Flight Time: τ Time Step: $\delta\tau = \tau_t - \tau_{t-1}$ Distance: x Height: h

Velocity: v Downrange Velocity: v_x Vertical Velocity v_x

Parameters

v_0 Muzzle Velocity (meters per second)

θ_0 Launch Angle (degrees)

Δ Retardation Coefficient $\{\Delta = (\frac{1}{2}\rho\frac{C_G}{BC})\}$

Initial Conditions

$v_{x,0} = v_0\cos(\theta_0)$ $v_{h,0} = v_0\sin(\theta_0)$ $x_0 = 0$ $h_0 = 0$

EXCEL Spreadsheet Entries

Column	Variable	Symbol	Cell Entry
A:	Time	τ	
B:	Distance	$x_t =$	$x_{t-1} + v_{x,t-1}\delta\tau$
C:	Height	$h_t =$	$h_{t-1} + v_{h,t-1}\delta\tau$
D:	x-Velocity	$v_{x,t} =$	$v_{x,t-1} + \Delta v_{t-1}v_{x,t-1}\delta\tau$
E:	h-Velocity	$v_{h,t} =$	$v_{x,t-1} + \Delta v_{t-1}v_{h,t-1})\delta\tau$
F:	Velocity	$v\ _t =$	$\sqrt{v_{x,t} + v_{h,t}}$
G:	$\frac{dv_x}{dt}$	$v_t v_{x,t} =$	$-\Delta v_{x,t}$
J:	$\frac{dv_h}{dt}$	$v_t v_{h,t} =$	$-9.8 - \Delta v_{h,t}$
K:	Flight Angle	$\theta_t =$	$arctan(\frac{h'}{x'})$
L:	Bullet Drop	$DROP_t =$	$x_0 tan(\theta_0) - h_t$

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